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Topological groupoids: II. Covering morphisms and G-spaces

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§ 1. Introduction

The theory of covering groupoids plays an important role in the applications of groupoids (cf. [11]), and in this theory there are two key results. One is that if G is a groupoid there is an equivalence between the category $\mathscr{O}p\,(G)$ of operations of G on sets (or G-sets as they are called) and the category $\mathscr{C}ov/G$ of covering groupoids of G. (This result seems to have been stated first in this form in [9], although the constructions involved had been used previously.) The other is that if G is a transitive groupoid[1]), there is a bijection between the equivalence classes of transitive covering groupoids of G and the conjugacy classes of subgroups of G. (This result is an abstract formulation of known results on covering spaces and the fundamental group – it is due to Higgins [11], page 110.)

The object of this paper is to prove topological versions of these results.

For the first result there is no problem. We define the category $\mathscr{T}\mathscr{C}ov/G$ of topological covering morphisms of the topological groupoid G; we follow Ehresmann [7] in defining the category $\mathscr{T}\mathscr{O}p\,(G)$ of G-spaces; and we prove the equivalence of these categories. This allows us to give a number of useful examples in these categories.

The second problem presents difficulties which are related to the fact that a transitive G-space need not be a homogeneous space of G. However we present a corresponding result for the locally trivial case.

In general this paper is independent of [5]; however, we will not repeat any of the results which appear there.

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§ 2. General Case

Recall that a morphism $q\colon H\to G$ of groupoids is a covering morphism if for each $y\in \mathrm{Ob}\,(H)$, the restriction of q mapping $St_H y\to St_G qy$ is a bijection. Let

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[1]) Transitive groupoids are also called connected groupoids in the literature.

$G \tilde{\times} \mathrm{Ob}\,(H)$ be given by the pullback diagram of sets.

$$\begin{tikzcd}
G \tilde{\times} \mathrm{Ob}\,(H) \arrow[r] \arrow[d] & \mathrm{Ob}\,(H) \arrow[d, "\mathrm{Ob}\,(q)"] \\
G \arrow[r, "\partial'"'] & \mathrm{Ob}\,(G)
\end{tikzcd}$$

If $q: H \to G$ is a covering morphism, we have a lifting function $s_q: G \tilde{\times} \mathrm{Ob}\,(H) \to H$ assigning to the pair (g, y) the unique element h of $St_H y$ such that $q(h) = g$; clearly s_q is inverse to $(q, \partial'): H \to G \tilde{\times} \mathrm{Ob}\,(H)$. So we can state: $q: H \to G$ is a covering morphism if and only if $(q, \partial'): H \to G \tilde{\times} \mathrm{Ob}\,(H)$ is a bijection. It is in terms of this function (q, ∂') that the notion of topological covering morphism is most conveniently phrased.

Definition. A $\mathcal{TG}$-morphism $q: H \to G$ of topological groupoids is a *topological covering morphism* if the function $(q, \partial'): H \to G \tilde{\times} \mathrm{Ob}\,(H)$ is a homeomorphism.

In such a case the inverse to (q, ∂') is written s_q and called the lifting function.

Notice that the identity $G \to G$ is a topological covering morphism. Also the composite of topological covering morphisms is again a topological covering morphism—this is one part of the following proposition

Proposition 1. *Suppose given a commutative diagram of morphisms of topological groupoids*

$$\begin{tikzcd}
H' \arrow[rr, "r"] \arrow[dr, "p"'] & & H \arrow[dl, "q"] \\
& G &
\end{tikzcd}$$

in which q is a topological covering morphism. Then p is a topological covering morphism if and only if r is a topological covering morphism.

The proof is given in § 4.

Let $\mathcal{TG}/G$ be the category of topological groupoids over G; the full subcategory of this on the topological covering morphisms is written $\mathcal{TC}ov/G$. By Proposition 1, the morphisms of this category are topological covering morphisms.

The basic examples of topological covering morphisms come from the relation between these and G-spaces, whose definition is due to Ehresmann [7].

Definition. A (left) G-space is a triple $(S; p, \varphi)$ where S is a topological space, $p: S \to \mathrm{Ob}\,(G)$ is a continuous function, and $\varphi: G \tilde{\times} S \to S$, $(a, s) \mapsto a \cdot s$, is a continuous action with $G \tilde{\times} S$ given by the pull-back diagram

$$\begin{tikzcd}
G \tilde{\times} S \arrow[r] \arrow[d] & S \arrow[d, "p"] \\
G \arrow[r, "\partial'"'] & \mathrm{Ob}\,(G)
\end{tikzcd}$$

The action φ must satisfy the usual axioms

(1) $p(a \cdot s) = \partial a$
(2) $b \cdot (a \cdot s) = (ba) \cdot s$
(3) $1_{p(s)} s \quad = s$

whenever these expressions are defined.

By an abuse of language, we also say that S is a (left) G-space via p.

A morphism of (left) G-spaces $(S, p, \varphi) \to (S', p', \varphi')$ consists of a continuous function $f \colon S \to S'$ such that $p'f = p$ and $f(a \cdot s) = a \cdot f(s)$ whenever $a \cdot s$ is defined. So we have a category $\mathcal{T}\mathcal{O}p\,(G)$ of G-spaces.

Theorem 2. *The categories $\mathcal{T}\mathcal{C}ov/G$ and $\mathcal{T}\mathcal{O}p\,(G)$ are equivalent.*

Proof. We define functors $\Gamma \colon \mathcal{T}\mathcal{C}ov/G \to \mathcal{T}\mathcal{O}p\,(G)$, $\Phi \colon \mathcal{T}\mathcal{O}p\,(G) \to \mathcal{T}\mathcal{C}ov/G$ and natural equivalences $\Gamma\Phi \cong 1$, $\Phi\Gamma \cong 1$.

Let $q \colon H \to G$ be a topological covering morphism, and let $s_q \colon G \mathbin{\bar{\times}} \mathrm{Ob}\,(H) \to H$ be the continuous inverse of (q, ∂'). Let $\varphi = \partial s_q \colon G \mathbin{\bar{\times}} \mathrm{Ob}\,(H) \to \mathrm{Ob}\,(H)$. Then φ is continuous and it is clear that $\Gamma(q) = (\mathrm{Ob}\,(H);\ \mathrm{Ob}\,(q), \varphi)$ is a G-space. A map of covering morphisms induces a morphism of G-spaces, so we have a functor as required.

Now let $(S;\ p;\ \varphi)$ be a (left) G-space. Then we make $G \mathbin{\bar{\times}} S$ into a topological groupoid with object space S as follows. The initial and final maps are defined by $\partial'(a, s) = s$, $\partial(a, s) = a \cdot s$; $\theta((b, t), (a, s)) = (b \cdot a, s)$; $u(s) = (1_{ps}, s)$; $\sigma(a, s) = (a^{-1}, a \cdot s)$. The verification that $G \mathbin{\bar{\times}} S$ is a topological groupoid is straight forward. The projection $q \colon G \mathbin{\bar{\times}} S \to G$ is a $\mathcal{T}\mathcal{G}$-morphism and it is clearly a topological covering morphism since $(q, \partial') \colon G \mathbin{\bar{\times}} S \to G \mathbin{\bar{\times}} S$ is the identity. Once again we have a functor Φ as required.

Clearly $\Gamma\Phi = 1$. The natural equivalence $\Phi\Gamma \cong 1$ is given by the fact that if $q \colon H \to G$ is a topological covering morphism, then $s_q \colon G \mathbin{\bar{\times}} \mathrm{Ob}\,(H) \to H$ is an isomorphism of topological groupoids.

Remark. It is clear from the above proof that there is a general result. Let $\mathscr{E}$ be an arbitrary category with pull-backs, and let G be a groupoid object in $\mathscr{E}$. A *covering groupoid* of G is a morphism $q \colon H \to G$ of groupoid objects in $\mathscr{E}$ such that $(q, \partial') \colon H \to G \mathbin{\bar{\times}} \mathrm{Ob}\,(H)$ is an isomorphism in $\mathscr{E}$. Then the categories $\mathscr{E}\text{-}\mathcal{O}p\,(G)$ and $\mathscr{E}\text{-}\mathcal{C}ov/G$ are equivalent.

We can now give examples of coverings and G-spaces by giving either first, and can also translate concepts from one category into the other.

Example 1. If G is a topological groupoid, then $\mathrm{Ob}\,(G)$ is a left G-space via the identity, the action being given by $a \cdot x = y$ for $x \in \mathrm{Ob}\,(G)$ and $a \in G(x, y)$. Note that this action derives from the identity covering morphism $1 \colon G \to G$.

If $(S;\ p, \varphi)$ is a G-space, then the *orbits* of S are the equivalence classes under the equivalence relation $s \sim t$ if and only if $t = a \cdot s$ for some a in G. These orbits can be identified with the (abstract) components of the groupoid $G \mathbin{\bar{\times}} S$ defined in the proof of Theorem 2. In particular, G operates transitively on S (i.e. S has only one orbit) if and only if the groupoid $G \bar{\times} S$ is transitive.

The set of orbits of S under this (left) action of G is written $G \backslash S$, and this set is given the identification topology with respect to the projection $S \to G \backslash S$. However, unlike in the case of groups, this projection need not be an open map.

Example 2. Let $f: S \to T$ be an identification map which is not an open map. Define a topological groupoid G by $\mathrm{Ob}\,(G) = S$, and $G(s, s') = \emptyset$ if $f(s) \neq f(s')$, and otherwise $G(s, s') = \{(s, s')\}$; thus G has its topology as a subset of $S \times S$. The unique composition in G which makes it a groupoid also makes it a topological groupoid. Then S, the object space of G, is a left G-space, and the orbit space $G \backslash S$ may be identified with T. So in this case $S \to G \backslash S$ is not open.

If $(S; p, \varphi)$ is a G-space, and $s \in S$, then the *stability group* (*or isotropy group*) of s is $G_s = \{a \in G : a \cdot s = s\}$. Clearly the covering morphism $q : G \bar{\times} S \to G$ maps the object group $(G \bar{\times} S)\{s\}$ isomorphically to G_s.

A topological covering morphism $q: H \to G$ is *regular* if for any $x \in \mathrm{Ob}\,(G)$ and $a \in G\{x\}$, then either all or none of the elements of $q^{-1}(a)$ lie in object groups of H. It is easily verified that q is regular if and only if for each $y \in \mathrm{Ob}\,(H)$, the group $q(H\{y\})$ is normal in $G\{qx\}$. Similarly, a G-space $(S; p, \varphi)$ is *regular* if for all s in S, the stability group G_s of s is normal in $G\{ps\}$.

In a similar way to left G-spaces we define a *right G-space* $(p, \varphi; S)$, where $p: S \to \mathrm{Ob}\,(G)$ is continuous, $\varphi : S \bar{\times} G \to S$, $(s, a) \mapsto s \cdot a$, is continuous with $S \bar{\times} G$ given by the pullback

$$\begin{tikzcd}
S \bar{\times} G \arrow[r] \arrow[d] & S \arrow[d, "p"] \\
G \arrow[r, "\partial'"'] & \mathrm{Ob}\,(G)
\end{tikzcd}$$

The axioms are that $p(s \cdot a) = \partial' a$, $s \cdot 1_{p(s)} = s$, $(s \cdot a) \cdot b = s \cdot (ab)$ whenever they are defined. The space of orbits of the action is again well-defined, and for a right G-space S is written S/G.

We can transfer left G-spaces to right G-spaces, in the same way as is done for actions of groups, by the rule $s \cdot a = a^{-1} \cdot s$.

We could also take our above definition of covering morphism to be a *left-covering* and define a right-covering $q: H \to G$ to be a morphism such that (∂, q): $H \to \mathrm{Ob}\,(H) \bar{\times} G$ is a homeomorphism. However it is easy to see that a $\mathcal{TG}$-morphism is a right-covering if and only if it is a left-covering.

Example 3. Let $p: X \to Y$ be a covering map of topological spaces, where X, Y are locally path-connected and semi-locally 1-connected. In [4] a "lifted topology" is described on the fundamental groupoids πX, πY so that they become topological groupoids, and it is proved that $\pi p: \pi X \to \pi Y$ is a topological covering morphism. So X obtains the structure of a πY-space.

Another example of covering morphisms comes from the morphism groupoid (X, G) used in [3].

Let X, G be topological groupoids. The set of morphisms $X \to G$ is written $M(X, G)$; we give this the compact-open topology (as a set of functions from the space of elements of X to the space of elements of G). We put a topology on the groupoid (X, G) so that it becomes a topological groupoid with object space $M(X, G)$.

If A, B are sets, let $F(A, B)$ denote the space of functions $A \to B$ with the compact-open topology. The elements of (X, G) can be taken as pairs (f, ϑ) such that $f(x) = \partial' \vartheta(x)$ for each $x \in \mathrm{Ob}\,(X)$ ([2] p. 196). Thus (X, G) can be identified with the pull-back in the diagram

$$\begin{tikzcd}
(X, G) \arrow[r, dotted] \arrow[d, dotted, "\partial'"'] & F(\mathrm{Ob}\,(X), G) \arrow[d, dotted, "\partial'_*"] \\
M(X, G) \arrow[r, dotted, "\mathrm{Ob}"] & F(\mathrm{Ob}\,(X), \mathrm{Ob}\,(G))
\end{tikzcd}$$

and this defines the topology on (X, G). The structure maps of (X, G) are: (i) ∂': $(X, G) \to M(X, G)$ as in the above diagram; (ii) ∂: $(X, G) \to M(X, G)$ is given by $(f, \vartheta) \mapsto g$, where $g(a) = (\vartheta \partial a) \circ (fa) \circ (\vartheta \partial' a)^{-1}$, $a \in X$; (iii) the identity u: $M(X, G) \to (X, G)$ is given by $u(f) = (f, \hat{f})$, where $\hat{f}: x \mapsto 1_{f(x)}$, $x \in \mathrm{Ob}\,(X)$; (iv) the inverse σ: $(X, G) \to (X, G)$ is given by $(f, \vartheta) \to (\partial(f, \vartheta), \vartheta^{-1})$. Clearly all these structure functions are continuous, and so (X, G) is a topological groupoid.

Proposition 3. *Let i: $A \to X$ be a morphism of topological groupoids such that $\mathrm{Ob}\,(i)$ is a homeomorphism. Then i^*: $(X, G) \to (A, G)$ is a topological covering morphism.*

Proof. Consider (i^*, ∂'): $(X, G) \to (A, G) \mathbin{\tilde{\times}} M(X, G)$. Now $(A, G) = M(A, G) \mathbin{\tilde{\times}} F(\mathrm{Ob}\,(A), G)$. So we define an inverse to (i^*, ∂') to be the composite

$$M(A, G) \mathbin{\tilde{\times}} F(\mathrm{Ob}\,(A), G) \mathbin{\tilde{\times}} M(X, G) \xrightarrow{\mathrm{proj}} F(\mathrm{Ob}\,(A), G) \mathbin{\tilde{\times}} M(X, G)$$
$$\xrightarrow{(\mathrm{Ob}(i))^{-1} \tilde{\times} 1} F(\mathrm{Ob}\,(X), G) \mathbin{\tilde{\times}} M(X, G)\,.$$

We can combine left and right actions in a way which has important applications.

Let G, H be topological groupoids. A $G-H$-*bispace* is a quintuple $(q, \psi; S; p, \varphi)$ where S is a topological space, $(S; p, \varphi)$ is a left G-space, $(q, \psi; S)$ is a right H-space and

$$\begin{aligned}
q(a \cdot s) &= q(s) \\
p(s \cdot b) &= p(s) \\
a \cdot (s \cdot b) &= (a \cdot s) \cdot b
\end{aligned}$$

whenever $s \in S$, $a \in G$, $b \in H$ and $a \cdot s$, $s \cdot b$ are defined. Notice that if $y \in \mathrm{Ob}\,(H)$ then the action of G makes $q^{-1}(y)$ a left G-space, while if $x \in \mathrm{Ob}\,(G)$ then the action of H makes $p^{-1}(x)$ a right H-space.

By an abuse of language we also say S is a $G-H$-bispace via $p-q$.

A standard example of $G-G$-bispace is the groupoid G itself via $\partial - \partial'$ with

left and right actions given by composition in G. In particular, if $x \in \mathrm{Ob}(G)$, then
$\mathrm{St}_G\, x$ … is a left G-space via t. These particular actions are important
what follows.

Example 1. One of the constructions in the proof of Theorem 2.
in $G-H$-bispaces. Let S be a $G-H$-bispace via $p-q$. Then we define
groupoid $G \ltimes S \rtimes H$ to have object space S and elements the
$G \times S \times H$ such that $p(s) = t(g)$, $q(s) = t(h)$. The initial and fin
by $i(g,s,h) = s$, $f(g,s,h) = g \cdot s \cdot h$; composition is
the unit and inverse functions are $u(s) = (1, s, 1)$ and
respectively. Now let $G \ltimes H$ be the subgroupoid of
there is an s in S with
jection morphism. Then
is a homeomorphism. So π is a topological
An important use of $G-H$-bispaces
spaces of H-orbits. Suppose S is a G
action of G on S defines a left action
is a difficulty in proving continuity
of identification maps need not
overcome in the useful special

Theorem 4. If S is a
is Hausdorff, then
structure of a left G
The proof is
Let G be
$G(x)$. Since
If
G_x

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conditions.
topological
es (g, s, h) in
maps are given
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of pairs (g, h) such that
$\rtimes H \to G \ltimes H$ be the pro-
S is $(g, s, h) \mapsto (g, h, s)$, and
ous morphism.
n constructing left actions of G on
-bispace via $p-q$. Although the left
on the set S/H, as is easy to check, there
n this action due to the fact that a pull-back
e an identification map, this difficulty can be
case given by the following theorem.

H-bispace in which H is a topological group and $\mathrm{Ob}(G)$
action of G on S determines on the orbit space S/H the
ace.

ven in §4.

topological groupoid, let $x \in \mathrm{Ob}(G)$ and let D be any subgroup of
is a $G-G$-bispace, it follows by restriction that $\mathrm{St}_G\, x$ is a $G-D$-bispace.
G_D to be the space $(\mathrm{St}_G\, x)/D$ of left cosets of D. It is easily verified that
T-space if and only if D is closed in $\mathrm{St}_G\, x$, and we will see later (Propo-
n 14) that this implies G_D is Hausdorff.

Corollary 5. If $\mathrm{Ob}(G)$ is Hausdorff, and D is a subgroup of G, then left multi-
plication gives G_D the structure of a left G-space.

This follows easily from Theorem 4.

Let G be a topological groupoid and S a transitive left G-space via $p: S \to \mathrm{Ob}(G)$. Let $s \in S$, and let D be the group of stability of s. Then the mapping $\chi_s: \mathrm{St}_G\, p(s) \to S$, $h \mapsto h \cdot s$ is a surjection and defines a continuous bijection $\varphi_s: G_D \to S$. In general φ_s will not be a homeomorphism—however, if φ_s is a homeomorphism for one s in S then φ_s will be a homeomorphism for each s in S, and in this case we call S a homogeneous space of G.

If G is a topological group, there are useful conditions for a G-space to be homogeneous. For example, if G is a topological group which is complete and satisfies the second axiom of countability, while S is a Hausdorff, non-meagre G-space, then S is homogeneous (cf. [7] and [18]). For the groupoid case, we give later a rather different type of condition for a G-space to be homogeneous.

Let $f\colon H \to G$ be any morphism of topological groupoids, and let $y \in \mathrm{Ob}\,(H)$; the *characteristic group* of f at y is the subgroup $f(H\{y\})$ of $G\{f(y)\}$. Now Corollary 5 can be restated in a way which shows it to be the basic existence theorem in the theory of topological covering morphisms.

Theorem 6. *Let G be a transitive topological groupoid with* HAUSDORFF *object space. Let $x \in \mathrm{Ob}\,(G)$, and let D be a subgroup of $G\{x\}$. Then there exists a topological covering morphism $q\colon H \to G$, such that H is transitive and there is a y in* Ob (H) *such that the characteristic group of q at y is D. Further $q\colon H \to G$ satisfies the following universal property:*

if $r\colon K \to G$ is a topological covering morphism, $z \in$ Ob (K) *is such that $r(z) = x$ and the characteristic group of r at z contains D, then there is a unique topological covering morphism $s\colon H \to K$ such that $s(y) = z$ and $rs = q$.*

The proof is given in § 4.

A special case of this theorem is when D is the trivial subgroup, so that $G_D = St_G x$. Then $G \ltimes St_G x \to G$ is called a *universal topological covering morphism* of G, since it covers any other topological covering morphism of G.

§ 3. The locally trivial case

An important class of topological groupoids introduced by EHRESMANN is that of locally trivial groupoids. These arise in nature because of their close connection with principal bundles. From our point of view they are convenient because one can construct continuous lifts of morphisms (Proposition 7).

Definition. A topological groupoid G is *locally trivial* if for each $x_0 \in \mathrm{Ob}\,(G)$ there is an open neighbourhood U_{x_0} of x_0 and a continuous function $\lambda_{x_0}\colon U_{x_0} \to G$ such that $\lambda_{x_0}(x) \in G(x_0, x)$ for all x in U_{x_0}. (There is clearly no loss in generality in assuming $\lambda_{x_0}(x_0) = 1_{x_0}$.)

The following example shows that local triviality is a restriction.

Example 5. Let G be a non-trivial topological group with the discrete topology, and let S be G with the indiscrete topology. Multiplication in the group turns S into a left G-space. The topological groupoid $G \mathbin{\widetilde{\ltimes}} S$ satisfies: $(G \mathbin{\widetilde{\ltimes}} S)\,(s, t)$ has exactly one element for all $s, t \in S$. However $G \mathbin{\widetilde{\ltimes}} S$ is not locally trivial, since the identity mapping $S \to G$ is not continuous.

Suppose now given a diagram of morphisms of groupoids,

$$\begin{tikzcd} & H \arrow[d, "q"] \\ F \arrow[r, "f"'] & G \end{tikzcd}$$

in which q is a covering morphism. Suppose further that F is transitive and that $y \in \mathrm{Ob}\,(H)$, $z \in \mathrm{Ob}\,(F)$ satisfy $q(y) = f(z)$. Then a necessary and sufficient condition for f to lift to a morphism $f^*: F \to H$ such that $f^*(z) = y$ is that $f(F\{z\})$ is a subgroup of $q(H\{y\})$ ([11], page 107). However that proof of sufficiency involves choosing a tree in F, and so cannot be expected to go over to the topological case without additional assumptions.

Suppose given a commutative diagram

$$\begin{tikzcd}
 & H \arrow[d, "q"] \\
F \arrow[ur, "f^*"] \arrow[r, "f"'] & G
\end{tikzcd}$$

such that F, G, H are topological groupoids, f is a $\mathcal{T}\mathcal{G}$-morphism, q is a topological covering morphism, and f^* is a $\mathcal{G}$-morphism.

Proposition 7. *If F is locally trivial then f^* is continuous, i.e. is a $\mathcal{T}\mathcal{G}$-morphism.*

The proof is given in § 4.

This proposition enables algebraic results given in [2], [11] to be translated into the topological case as follows.

Let $q: H \to G$, $q': H' \to G$ be topological covering morphisms such that H, H' are transitive let $y \in \mathrm{Ob}(H)$, $y' \in \mathrm{Ob}(H')$ be such that $q(y) = q'(y')$, and let $C = q(H\{y\})$, $C' = q'(H'\{y'\})$.

Corollary 8. *If H is locally trivial, then there is a unique topological covering morphism $r: H \to H'$ such that $q'r = q$ and $r(y) = y'$ if and only if $C \subseteq C'$. Further if H' also is locally trivial then r is a topological isomorphism if and only if $C = C'$.*

Corollary 9. *Let H be a locally trivial topological covering groupoid of G such that $H(x, y)$ has one element for all objects x, y of H. Then H is a universal topological covering groupoid of G.*

To complete the story we need criteria for a topological covering of G to be locally trivial–this is most conveniently phrased in terms of G-spaces as in the next proposition. Theorem 12 then gives a useful condition for the existence of locally trivial topological coverings of G.

Let $f: X \to Y$ be a map of topological spaces. We say f is a *submersion* if for each $x \in X$ there is an open neighbourhood U of $f(x)$ and a continuous function $\lambda: U \to X$ such that $\lambda f(x) = x$ and $f\lambda = 1_U$.

Notice that a submersion is necessarily an open mapping.

Proposition 10. *Let G be a topological groupoid and $(S; p, \varphi)$ a left G-space. Then the following conditions are equivalent.*

(i) *$G \ltimes S$ is locally trivial.*

(ii) *For each $s \in S$, the function $\varphi_s: St_G p(s) \to S$, $a \mapsto a \cdot s$ is a submersion.*

(iii) *For each s in S the orbit G_s of s is open in S, and the projection $q_s: St_G p(s) \to Gs$ is a locally trivial principal G_s-bundle.*

The proof is given in § 4.

A consequence of Proposition 10 is that if S is a transitive G-space and $G \ltimes S$ is locally trivial, then S is homogeneous. Also by taking $S = \mathrm{Ob}(G)$ we see that

Proposition 10 contains a result of PRADINES [7], that if G is locally trivial and $x \in \mathrm{Ob}(G)$, then $t: St_G x \to \mathrm{Ob}(G)$ is a locally trivial principal $G\{x\}$-bundle.

A result in a different direction is the following.

Proposition 11. Let G be a transitive, locally trivial topological groupoid, (i) If $q: H \to G$ is a topological covering morphism such that H is transitive then $q: \mathrm{Ob}(H) \to \mathrm{Ob}(G)$ is an open map.

(ii) If $(X, p, \cdot)$ is a transitive left G-space then $p: X \to \mathrm{Ob}(G)$ is an

By results of §2 either of (i) or (ii) implies the other. The proof in §4.

The condition of local triviality cannot be dropped. use as in Example 5 then $\mathrm{Ob}\, q: \tilde{X} \to X$ is a (G)-space continuous and bijective. But q is not a homeomorphism

Let H be a topological group. A subgroup D of action of D on H makes H a locally trivial, sufficient condition for this is that the prope page 48, Corollary 8.3). For example, grds.

Recall that if G is a topological of $G\{x\}$, then $G_{x,D}$ is the space of left then $G_{x,D}$ is a left G-space. We say of $G\{x\}$.

Theorem 12. Let G be HAUSDORFF. Let D be

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then G, $\mathrm{Ob}\, q: X \to X$ is and so is not open.

said to be good if the right bundle; a necessary and $H \to H/D$ is a submersion [9], good subgroup of a LIE group is

groupoid, $x \in \mathrm{Ob}(G)$, and D is a subgroup costs of D, and that if $\mathrm{Ob}(G)$ is HAUSDORFF is a good subgroup of G if it is a good subgroup

locally trivial topological groupoid such that $\mathrm{Ob}(G)$ is good subgroup of G. Then $G_{x,D}$ is locally trivial.

given in §4.

be combined with previous results and the classification abstract case to yield a classification theorem for topo morphisms.

13. Let G be a locally trivial, transitive topological groupoid with object space, and such that all its closed subgroups are good. Then there bijection between conjugacy classes of closed subgroups of G and equivalence classes of locally trivial, transitive, topological coverings of G with HAUSDORFF object space.

This theorem follows from previous results and the following proposition, whose proof is given in §4.

Proposition 14. Let G be a locally trivial topological groupoid with HAUSDORFF object space. Let D be a subgroup of $G\{x\}$. Then the following conditions are equivalent:

(i) D is closed in $G\{x\}$;

(ii) $G_{x,D}$ is T_1;

(iii) $G_{x,D}$ is HAUSDORFF.

Finally, in "Topological Groupoids I" [3] we proved the existence of quotient topological groupoids. We now show that the quotient topological groupoid G/N

of a locally trivial topological groupoid G by a wide, totally disconnected, normal subgroupoid N is locally trivial. In fact we prove more, namely

Theorem 15. *Let G be a locally trivial, topological groupoid, and N a wide normal subgroupoid of G such that* $\mathrm{Ob}\,(p)\colon \mathrm{Ob}\,(G) \to \mathrm{Ob}\,(G/N)$ *is a submersion. Then G/N is a locally trivial topological groupoid.*

The proof is given in § 4.

§ 4. Proofs

Proof of **Proposition 1.** We have a commutative diagram

$$\begin{tikzcd}
H' \arrow[rr, "r"] \arrow[dr, "p"'] & & H \arrow[dl, "q"] \\
& G &
\end{tikzcd}$$

Let $q\colon H \to G$ and $r\colon H' \to H$ be topological covering morphisms. Then clearly, for each $h' \in \mathrm{Ob}\,(H')$, $p \mid St_{H'} h'$ is a homeomorphism onto $St_G p(h')$ so that p is an abstract covering morphism and the unique lifting morphism $s_p\colon G \tilde{\times} \mathrm{Ob}\,(H') \to H'$ exists. Continuity follows easily from the diagram,

$$\begin{tikzcd}
G \tilde{\times} \mathrm{Ob}\,(H') \arrow[r, dashed, "s_p"] \arrow[d, dashed, "(1 \tilde{\times} \mathrm{Ob}\,(r){,}\, \pi_2)"'] & H' \arrow[d, dashed, "s_r"] \\
G \tilde{\times} \mathrm{Ob}\,(H) \tilde{\times} \mathrm{Ob}\,(H') \arrow[r, dashed, "s_q \tilde{\times} 1"] & H \tilde{\times} \mathrm{Ob}\,(H')
\end{tikzcd}$$

where $\pi_2\colon G \tilde{\times} \mathrm{Ob}\,(H') \to \mathrm{Ob}\,(H')$ is the obvious projection.

Conversely, let $q\colon H \to G$ and $p\colon H' \to G$ be topological covering morphisms. Then again it is easy to see that for any $h' \in \mathrm{Ob}\,(H')$, $r \mid St_{H'} h'$ is a homeomorphism onto $St_H r(h')$ so that the unique lifting morphism $s_r\colon H \tilde{\times} \mathrm{Ob}\,(H') \to H'$ exists. Continuity follows from the diagram

$$\begin{tikzcd}
H \tilde{\times} \mathrm{Ob}\,(H') \arrow[rr, "s_r"] \arrow[dr, "q \tilde{\times} 1"'] & & H' \arrow[dl, "s_p"] \\
& G \tilde{\times} \mathrm{Ob}\,(H') &
\end{tikzcd}$$

Proof of Theorem 4. We are considering the $G-H$-bispace [illegible]. We have to prove that the action [illegible] induced by φ is continuous.

Let $r: S \to S/H$ be the quotient mapping. Then r is an open mapping, hence [illegible] is open, and so a quotient mapping. But [illegible] saturated subset of [illegible] and is also, since [illegible] is Hausdorff, closed in [illegible] is a quotient mapping. Since [illegible] it follows φ is continuous.

Proof of Theorem 6. By Corollary 5, [illegible] is a left G-space and [illegible] be the corresponding topological covering morphism. Let [illegible] be [illegible]. Then [illegible] the group of stability of [illegible] in [illegible].

Now suppose [illegible] given as above [illegible] and so we can define [illegible] factorises through [illegible]. Clearly [illegible] Theorem 2 a morphism [illegible] such that [illegible] By Proposition 1, [illegible] is a topological covering [illegible] and uniqueness is simple to verify.

Proof of Proposition 7. Let [illegible] then [illegible] to prove [illegible] continuous on [illegible]. Let [illegible]. Then

[illegible]

which is clearly [illegible]

Proof [illegible] exists an [illegible]

[illegible] be a trivialising cover of [illegible]. Let [illegible] is an open cover of [illegible] and so it is sufficient.

[illegible] continuous function of u.

Proposition 10. (i)⇒(ii) Let [illegible]. Then by hypothesis there [illegible] open neighbourhood U of [illegible] and a continuous function [illegible] [illegible] whenever [illegible]. Let [illegible] so that [illegible] and [illegible] for [illegible], while [illegible] we have [illegible] and [illegible]. (ii)⇒(i) Let [illegible], and suppose [illegible] is a submersion, and [illegible] is a continuous function from an open neighbourhood U of x such that [illegible] and [illegible]. Let [illegible] be given by [illegible]. Then [illegible] so [illegible] is locally trivial. (ii)⇒(iii) Let [illegible], and let [illegible] be a local cross-section of [illegible] such that [illegible]. Then U is contained in the orbit [illegible] of x, thus [illegible] is open in S. Now [illegible] is clearly a principal [illegible]-bundle in the sense of [illegible]. Hence it is also a principal [illegible]-bundle. But a principal bundle under a group action is locally trivial if and only if the projection to the orbit space is a submersion. This proves (iii). (iii)⇒(ii) Let [illegible]. If [illegible] is a locally trivial principal [illegible]-bundle then it is a submersion. Since [illegible] is open in S, it follows that [illegible] is a submersion.

Proof **of Proposition 11(ii).** Let $s \in S$. The following diagram is commutative. Since G is transitive and locally trivial, Proposition 10 implies ∂ is an open map. Since S is a transitive G-space, φ_s is surjective. Hence p is an open mapping.

$$\begin{tikzcd}
St_G\, p(s) \arrow[dr, "\varphi_s"] \arrow[dd, "\partial"'] & \\
 & S \arrow[dl, "p"] \\
\mathrm{Ob}\,(G) &
\end{tikzcd}$$

Proof **of Theorem 12.** We need the following lemma which is probably well-known.

Lemma. *Let H be a topological group and X a locally trivial principal H-bundle over X/H. Suppose that D is a subgroup of H such that $H \to H/D$ is a locally trivial principal D-bundle. Then $X \to X/D$ is a locally trivial principal D-bundle.*

Proof. Let $p: X \to X/H$, $q: H \to H/D$, $r: X \to X/D$ be the projections. Let $x \in X$; we wish to find a neighbourhood W of $r(x)$ such that $r^{-1}(W)$ is D-isomorphic to $W \times D$.

Let U be a neighbourhood of $p(x)$ such that there is an H-isomorphism α: $p^{-1}(U) \to U \times H$. Let $\alpha(x) = (p(x), x')$. Let V be a neighbourhood of $q(x')$ such that there is a D-isomorphism $\beta: q^{-1}(V) \to V \times D$. Then $W' = \alpha^{-1}(U \times q^{-1}(V))$ is an open neighbourhood of x which is D-isomorphic to $U \times V \times D$. Let $W = r(W')$. Then W is homeomorphic to $U \times V$, and $r^{-1}(W)$ is D-isomorphic to $W \times D$. This proves the lemma.

We now obtain Theorem 12, which is equivalent to $St_G x_0 \to G_D$ being a locally trivial principal D-bundle, by taking $X = St_G x_0$ and $H = G\{x_0\}$.

Proof **of Proposition 14.** Clearly (iii) ⇒ (ii) ⇒ (i). So we need only prove (i) ⇒ (iii). Now $\partial: St_G x_0 \to \mathrm{Ob}\,(G)$ is a locally trivial principal $G\{x_0\}$-bundle. By [12], page 71, Theorem 1.1, $\partial_D: G_D \to \mathrm{Ob}\,(G)$ is canonically isomorphic to the associated fibre bundle, with fibre $G\{x_0\}/D$. Therefore $\partial_D: G_D \to \mathrm{Ob}\,(G)$ is a locally trivial fibre bundle. But $\mathrm{Ob}\,(G)$ is HAUSDORFF, and $G\{x_0\}/D$ is HAUSDORFF, since D is closed in $G\{x_0\}$. Hence G_D is HAUSDORFF.

Proof **of Theorem 15.** Let $p: G \to G/N$ be the canonical $\mathfrak{T}\mathfrak{G}$-morphism. Let $C \in \mathrm{Ob}\,(G/N)$, and let $x \in \mathrm{Ob}\,(G)$ satisfy $p(x) = C$. Then, since $\mathrm{Ob}\,(p)$: $\mathrm{Ob}\,(G) \to \mathrm{Ob}\,(G/N)$ is a submersion, there is an open neighbourhood V of C and a continuous function $\mu: V \to \mathrm{Ob}\,(G)$ such that $\mu(C) = x$ and $\mathrm{Ob}\,(p)\,\mu = 1_V$.

Now G is locally trivial, so there is an open neighbourhood U of x and a continuous function $\lambda: U \to G$ such that $\lambda(x) = 1_x$ and $\partial\lambda = 1_U$. Let $U' = \mu^{-1}(U)$. Then U' is an open neighbourhood of C, and $\lambda' = p\lambda\mu: U' \to G/N$ satisfies $\partial\lambda' = 1_{U'}$ as required.

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Added in Proof:

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Correction: On p. 273 of [5] the first four references should be to A13, A32, A54 and A29, while on p. 280 the reference to A13 should be to A5.

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